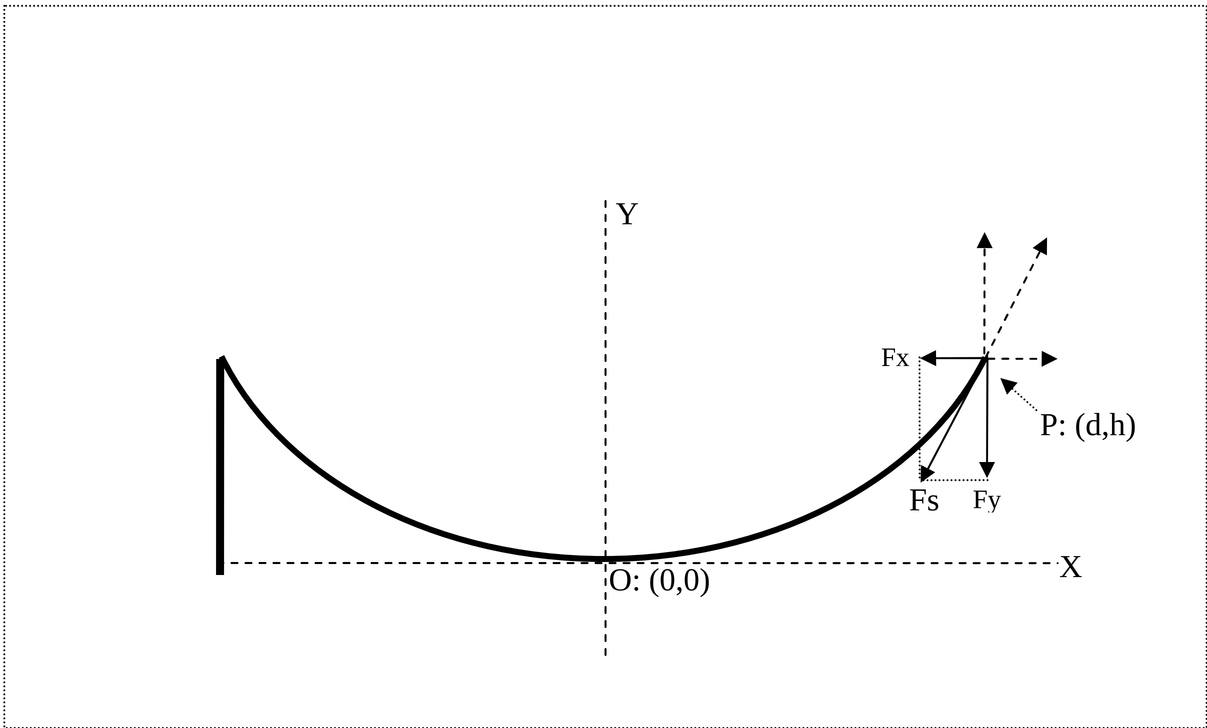
About the curve of a free hanging rope.



The aim of this document is to determine which function describes the curve of a rope hanging (in a steady state) between two fixed points. In order to make this possible we need to assume a few things about the rope.
First of all the rope needs to be inelastic and completely flexible. Also we assume that the rope has a uniform mass such that we can determine the mass of a segment of the rope by multiplying the length of the segment s with a constant ρ, where ρ is the mass per unit length of rope. In other words $m = \rho.s$. Further more the only external force working on the rope we'll examine will be gravity which is pointing down in the y direction. Since we are free to choose a coordinate system we choose it such that the lowest point of the rope's curve will be positioned at $x = 0 \wedge y = 0$ which we will call point O.
It should be obvious that at any point of the rope the tangent to the rope's curve equals the tangent to the forces which make up the tension in the rope because the rope will line up with the forces (it's flexible). This gives us a relation between the forces and the slope of the curve:

$$\boxed{\frac{Fy}{Fx} = \frac{dy}{dx} = y'} \quad (1)$$

Likewise it is easy to see that the curve should be symmetrical in the y axis (assuming of course that gravity works the same on both sides of the y axis). In this document we will look what happens on the right side of the drawing ($x \geq 0$) and we'll get the left side for free. Now since the rope is flexible and there is no active force working in the x direction, the only force working in the x direction is due to the tension in the rope which is a reactive force, the tension force in the x direction must therefore be equal in all parts of the rope. You may look at it like this; the rope will simply keep changing its shape until this condition is met. An other important thing to notice is that at point O there is no tension in the y direction, the tangent to the curve is 0. Therefore the tension in this point does not contribute to lifting the rope, so it should be obvious that the tension force in the y direction in any point P on the rope must be responsible for lifting the entire mass of the rope segment extending from point O to point P. This gives us the formula:

$$\boxed{Fy = \rho.g.s} \quad (2)$$

Of course g is the gravitational acceleration constant. This formula must hold for any point P on the rope which is separated by an arc length s from O. From simple calculus we know that the arc length of a line segment extending from $x = a$ to $x = b$ can be calculated using $s = \int_a^b \sqrt{1 + y'^2}\,dx$ (see notes at bottom of the document) combining this with (2) gives $Fy = \rho.g\int_0^x \sqrt{1 + y'^2}\,dx$ which combined with (1) results in the differential equation:

$$\boxed{\mu\int_0^x \sqrt{1 + y'^2}\,dx = y' \text{ with } \mu = \frac{\rho.g}{Fx}} \quad (3)$$

After first differentiating, then squaring followed by some shifting this leads to:

$$\boxed{y''^2 - \mu^2 y'^2 = \mu^2} \quad (4)$$

At first this might look a bit difficult to solve, but μ is a constant and might also be one. In which case the equation becomes quite easy. $y''^2 - y'^2 = 1$, now if we choose the function such that $y' = \sinh(x) \Rightarrow y'' = \cosh(x)$ our equation becomes $\cosh^2(x) - \sinh^2(x) = 1$ which is always true. So if μ equals one the function $y = \cosh(x + \varphi) + c$ is a valid solution. Well, let's see if we can get this working for other values of μ by making the solution a little more general like $y = \alpha.\cosh(\beta.x + \varphi) + c$. If this function is the solution we might as well assume that $\varphi = 0$ since $\cosh(x)$ has its minimum at $x = 0$ which is exactly what we chose for our rope. The differential equation (4) now becomes $\alpha^2\beta^4\cosh^2(\beta.x) - \alpha^2\beta^2\mu^2\sinh^2(\beta.x) = \mu^2$ which can only be true if $\alpha^2\beta^4 = \alpha^2\beta^2\mu^2 \Rightarrow \beta^2 = \mu^2$ because we need a common factor in order to get a constant outcome. We now have $\alpha^2\mu^4\left(\cosh^2(\beta.x) - \sinh^2(\beta.x)\right) = \alpha^2\mu^4 = \mu^2 \Rightarrow \alpha^2 = \mu^{-2}$ so this means that the function will be a solution if and only if $\beta = \pm\mu \wedge \alpha = \pm\mu^{-1}$. It doesn't matter which sign we choose for β because the function is symmetrical in the y axis, so let's choose it positive. From the shape of the curve we can see that α must be positive for a hanging rope so all things combined give us the solution $y = \mu^{-1}\cosh(\mu.x) + c$. Having chosen $y = 0$ at $x = 0$ and after expanding μ we find our final solution.

$$\boxed{y = \frac{Fx}{\rho.g}\cosh\left(\frac{\rho.g}{Fx}x\right) - \frac{Fx}{\rho.g}} \quad (5)$$

This curve is often called a catenary.

Well we have the curve, but isn't it odd that the curve seems dependent on the density of the wire? Our (or at least mine) intuition tells us that it shouldn't be. Let's have a closer look. We will use μ again as defined in (3). Now we will examine a point P with coordinates (d, h) at which the rope is fixed. The s in formula (2) is the length of the rope between points O and P, so let's call it L instead. From (5) we see $h = \mu^{-1}\cosh(\mu.d) - \mu^{-1} \Rightarrow \quad \mu.h + 1 = \cosh(\mu.d)$. Now from (1), (2), (5) it follows that $Fy = \rho.g.L = Fx.y' = Fx\sinh(\mu.d) \Rightarrow \quad \mu.L = \sinh(u.d)$. But, this means that $(\mu.h + 1)^2 - (\mu.L)^2 = 1$ since $\cosh^2(\mu.d) - \sinh^2(\mu.d) = 1$. Solving this equation easily gives us

$$\boxed{\mu = \frac{2h}{L^2 - h^2}} \quad (6)$$

So the curve is indeed independent of the density of the rope since we can determine μ from L and h without any density or mass property of the rope.

(5) becomes: $y = \frac{L^2 - h^2}{2h}\cosh\left(\frac{2h}{L^2 - h^2}x\right) - \frac{L^2 - h^2}{2h}$ (7) for the curve of the rope between points O and P.

Finally using (6) we can write $h = \frac{L^2 - h^2}{2h}\cosh\left(\frac{2h}{L^2 - h^2}d\right) - \frac{L^2 - h^2}{2h}$ so after some agony we get $d = \frac{L^2 - h^2}{2h}\operatorname{acosh}\left(\frac{L^2 + h^2}{L^2 - h^2}\right)$ which after some more struggle can also be written as:

$$d = \frac{L^2 - h^2}{2h}\ln\left(\frac{L + h}{L - h}\right) \quad (8)$$

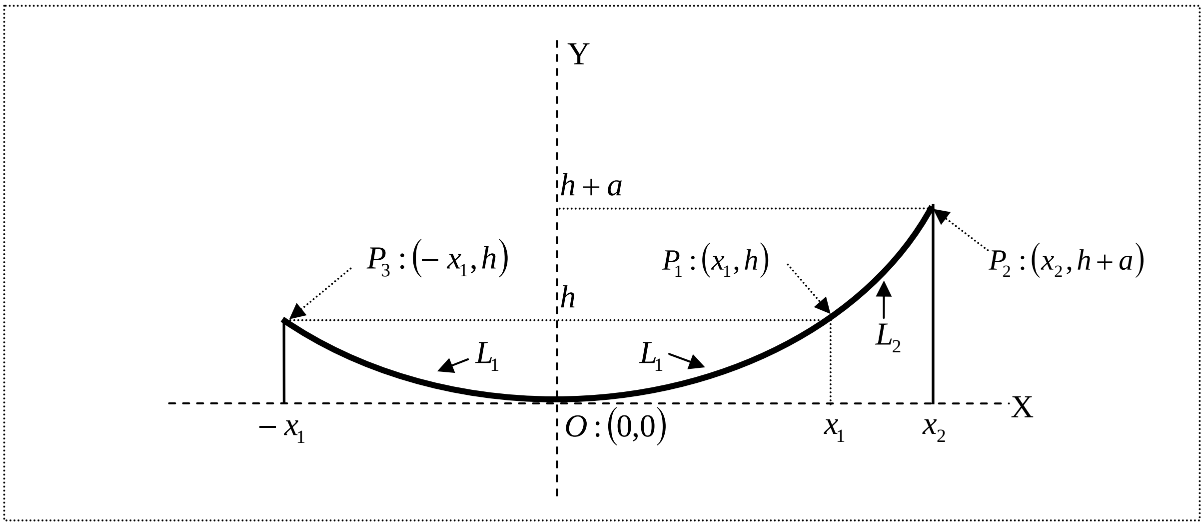
This gives us the means to calculate d from h and L. To my knowledge there is no algebraic way to calculate L from d and h, or h from d and L. These calculations have to be done numerically.

A list of the forces in P:

$$Fx = \frac{\rho.g\left(L^2 - h^2\right)}{2h}, \quad Fy = Fx.\sinh(\mu.d) = \rho.g.L, \quad Fs = Fx.\cosh(\mu.d) = \frac{\rho.g\left(L^2 + h^2\right)}{2h}$$

See next page for example of a rope attached between to points of unequal heights.

Rope attached between to 2 points of unequal heights.



In the drawing above you can see a rope fixed at 2 ends p_3 and p_2 with different heights. In this drawing h represents the height of p_3 above the lowest point of the curve. If we call d the horizontal distance between points p_3 and p_2 you can see that $d = x_1 + x_2$ and if the total length of the rope is called L it follows that $L = L_2 + 2L_1$. Also a is the height difference between the points.

Let's write (5) again as $y = \mu^{-1}\cosh(\mu.x) - \mu^{-1}$. Applied on the drawing above this gives us

$a = \mu^{-1}(\cosh(\mu.x_2) - \cosh(\mu.x_1)) = 2\mu^{-1}\sinh\left(\frac{\mu\cdot(x_2+x_1)}{2}\right)\sinh\left(\frac{\mu\cdot(x_2-x_1)}{2}\right)$. Also

$L_2 = \mu^{-1}(\sinh(\mu.x_2) - \sinh(\mu.x_1)) = 2\mu^{-1}\sinh\left(\frac{\mu\cdot(x_2-x_1)}{2}\right)\cosh\left(\frac{\mu\cdot(x_2+x_1)}{2}\right)$. After dividing these 2 and substituting μ using (6) it follows that:

$$\boxed{\frac{a}{L_2} = \frac{a}{L-2L_1} = \tanh\left(\frac{\mu\cdot(x_2+x_1)}{2}\right) = \tanh\left(\frac{\mu\cdot d}{2}\right) = \tanh\left(\frac{h\cdot d}{L_1^2 - h^2}\right)} \quad (9)$$

From (6) it also follows that $\mu = \frac{2h}{L_1^2 - h^2} = \frac{2(h+a)}{(L-L_1)^2 - (h+a)^2}$. Solving this equation for L_1

Gives us: $$\boxed{L_1 = -\frac{h\cdot L \pm \sqrt{h\cdot(a+h)\cdot(L^2-a^2)}}{a}} \quad (10).$$

As you can see this equation has 2 solutions, the $\pm$ sign should be chosen '-' if the lowest fixed point is left of the Y axis (P_3) and '+' if this point is right of the Y axis (P_1).

To determine on which side of the Y axis the lowest fixed point is located we can use (8).

If $\frac{L^2-a^2}{2a}\ln\left(\frac{L+a}{L-a}\right) > d$ the point is located on the left side and we should use the - sign.

Substituting L_1 from (10) into (9) gives:

(11)

$$d = \frac{(L^2-a^2)\cdot(a+2h)\pm 2L\cdot\sqrt{h\cdot(a+h)\cdot(L^2-a^2)}}{a^2}\cdot \operatorname{atanh}\left(\frac{a^2}{L\cdot(a+2h)\pm 2\sqrt{h\cdot(a+h)\cdot(L^2-a^2)}}\right)$$

Again the $\pm$ sign before each root should be chosen negative if the lowest fixed point is left of the Y axis and positive if it is on the right side of the Y axis.

Example 1:

The simplest example will be one where we need to calculate the required distance between two fixed points if the length of the rope and the height of all points are known, this can be done using a simple calculator. Let's assume we have 2 poles with heights $h_1 = 10\text{m}$ and $h_2 = 15\text{m}$, we have a rope with a length of $L = 30\text{m}$ and we want the deepest point of the rope to be 8m above ground. The deepest point above ground is of course $h_1 - h$ so the h in our equation is $h = 10\text{m} - 8\text{m} = 2\text{m}$. Now all we need to do is fill in the known values in equation (11), this gives us (using '-' for the $\pm$ sign):

$d = \frac{875\cdot 9 - 60\cdot\sqrt{2\cdot 7\cdot 875}}{25}\cdot \operatorname{atanh}\left(\frac{25}{30\cdot 9 - 2\sqrt{2\cdot 7\cdot 875}}\right) = 28.05$. We can now find all parameters of the catenary using the formulas in this document. Using (10) we calculate $L_1 = 10.14$, $\mu = \frac{2h}{L_1^{\,2} - h^2} = 0.0405$. $x_1 = \frac{\operatorname{asinh}(\mu\cdot L_1)}{\mu} = 9.87\text{m}$ where x_1 represents the horizontal distance from the deepest point to the lowest pole. The catenary curve becomes:

$$y = \mu^{-1}\cosh(\mu\cdot x - \mu\cdot x_1) + k \Rightarrow y = 24.7\cdot\cosh(0.0405\cdot x - 0.400) - 16.7$$

Where y is the height of the rope above ground and x is the distance from the lowest pole in the direction of the higher pole. Since at $x = 0,\ y = h_1$ it's obvious that $k = h_1 - h - \mu^{-1}$.

Example 2:

In a physical situation where the length of the rope, the horizontal distance between the points and the height of the points is know, (11) gives us one equation with one unknown, h which can only be solved numerically. Let's assume we have two poles at a distance of $d = 20\text{m}$ from each other, the poles have heights of $h_1 = 10\text{m}$ and $h_2 = 15\text{m}$, between the poles hangs a rope with a length of $L = 28\text{m}$. Now let's calculate the curve of the rope. The lowest fixed point is left of the Y axis so we use '-' in equation (11) to get:

$20 = \frac{759\cdot(5+2h) - 2L\cdot\sqrt{h\cdot(5+h)\cdot 759}}{25}\cdot \operatorname{atanh}\left(\frac{25}{28\cdot(5+2h) - 2\sqrt{h\cdot(5+h)\cdot 759}}\right)$, we now solve h from this equation numerically (using any suited program, see last page of document for a C example program), it turns out $h = 6.21\text{m}$. Having calculated h we can easily find all parameters required for the catenary curve.

$L_1 = 11.2\text{m},\ \mu = 0.143,\ x_1 = 8.74\text{m}$, deepest point is $h_1 - h = 3.79\text{m}$ above the ground. The catenary curve as seen from the lowest pole becomes:

$$y = \mu^{-1}\cosh(\mu.x - \mu.x_1) + k \Rightarrow y = 6.99\cosh(0.143x - 1.25) - 3.2$$

Example 3:

Example for a curve with the lowest fixed point on the right side of the Y axis. Given the values for $h_1 = 10\text{m}, h_2 = 20\text{m}, d = 11\text{m}$ and $L = 15\text{m}$. We numerically solve h from (11) (Note that we should use '+' now for each $\pm$ in the equation):

$11 = \frac{125\cdot(10+2h)+30\cdot\sqrt{h\cdot(10+h)\cdot 125}}{200}\cdot \text{atanh}\left(\frac{100}{15\cdot(10+2h)+2\sqrt{h\cdot(10+h)\cdot 125}}\right)$, this gives us $h = 2.17$. So the relevant variables are: $L_1 = -9.0\text{m}, \mu = 0.057, x_1 = -8.65\text{m}$. The curve for this catenary as seen from the lowest pole (note the lowest point is of course $h_1 = 10\text{m}$) becomes:

$$\boxed{y = \mu^{-1}\cosh(\mu.x - \mu.x_1) + k \Rightarrow y = 17.6\cosh(0.057x + 0.49) - 9.75}$$

As you can see in this example $L_1 < 0$. Physically this means that L_2 in our equations is now represented by the entire length of the rope in the drawing and L in our equations is represented by L_2 in the drawing (which is of course exactly what we wanted).

Example 4:

Finally we can also have a situation where the distance between two fixed points, say $d = 30\text{m}$, as well as the height of each point, say $h_1 = 12\text{m}$ and $h_2 = 18\text{m}$, is known and we require the lowest point of the rope to be at 8m, $h_1 - h = 8\text{m} \Rightarrow h = 4\text{m}$. We want to know the length of the rope needed to satisfy these requirements. Again we use (11) to get:

$30 = \frac{(L^2-36)\cdot 14 - 2L\cdot\sqrt{40\cdot(L^2-36)}}{36}\cdot \text{atanh}\left(\frac{36}{L\cdot 14 - 2\sqrt{40\cdot(L^2-36)}}\right)$. Numerically solving this equation gives $L = 34.12\text{m}$, so $L_1 = 12.66\text{m}, \mu = 0.0555, x_1 = 11.8\text{m}$. The catenary curve becomes:

$$\boxed{y = \mu^{-1}\cosh(\mu.x - \mu.x_1) + k \Rightarrow y = 18\cosh(0.0555x - 0.654) - 10}$$

Again x is the distance from the lowest pole in the direction of the higher pole, y is the height above the ground.

Some notes:

By definition: $\boxed{\cosh(x) = \frac{e^x + e^{-x}}{2}}$ $\boxed{\sinh(x) = \frac{e^x - e^{-x}}{2}}$ $\boxed{\text{atanh}(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right)}$

To see that $s = \int_a^b \sqrt{1+y'^2}\,dx$ consider the method below. Where ds is an infinitesimally small piece of the curve in which the tangent is $\frac{dy}{dx}$. Not exactly the proper way, I admit but I assure you it works.

$$\boxed{ds = \sqrt{dx^2 + dy^2} \Rightarrow s = \int\sqrt{dx^2 + dy^2} = \int\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx = \int\sqrt{1+y'^2}\,dx}$$

This document was written by Ruud v Gessel April 2001. And modified October 2007. If there are any errors in the doc or you have any remarks regarding its contents you may contact me.

Small C program to numerically calculate L or h from (11) using MSVC

```
#include "stdafx.h"
#include "math.h"

#define MAXERR 1e-10          // Absolute precision of calculation
#define MAXIT 100             // Maximum iterations (will never reach 100 unless an error has occurred)
#define TV ((upper+lower)/2)  // Test value for our iteration routines, gives the middle of the solution range

double atanh(double x)                                    // Not defined in math library arctanh
{
    return 0.5*log((1+x)/(1-x));                          // Return atanh(x)
}

double Calc_D(double a, double L, double h, double sgn)   // Calculates d from equation 11
{
    double q=2*sgn*sqrt(h*(a+h)*(L*L-a*a));               // + or - 2* the root used in (11)
    return ((L*L-a*a)*(a+2*h)-L*q)/(a*a)*atanh(a*a/(L*(a+2*h)-q));  // return calculated d from eq (11)
}

double Solve_h(double a, double L, double d)              // Routine to solve h from a, L and d
{
    int n=1;                                              // Iteration counter (quit if >MAXIT)
    double s=((L*L-a*a)/(2*a)*log((L+a)/(L-a))<d) ?-1:1;  // Left or right of Y axis ?
    double lower=0, upper=(L-a)/2;                        // h must be within this range

    while((upper-lower) > MAXERR && (++n)<MAXIT)          // Until range narrow enough or MAXIT
        if(Calc_D(a,L,TV,s)*s<d*s) upper=TV; else lower=TV;  // Narrow the range of possible h

    printf("Found h=%3.10f after %d iterations.\n\r",TV,n);  // If you see 100 iterations ➔ error
    return s*TV;                                          // Returns h (- signals right of Y axis)
}

double Solve_L(double a, double h, double d)              // Routine to solve L from a, h and d
{
    int n=1;                                              // Iteration counter (quit if >MAXIT)
    double lower=sqrt((d*d+a*a)), upper=2*h+d+a;          // L must be within this range

    while((upper-lower) > MAXERR && (++n)<MAXIT)          // Until range narrow enough or MAXIT
        if(Calc_D(a,TV,h,1)>d) upper=TV; else lower=TV;   // Narrow the range of possible L

    printf("Found L=%3.10f after %d iterations.\n\r",TV,n);  // If you see 100 iterations ➔ error
    return TV;                                            // Returns L
}

int main(int argc, char* argv[])
{
    h=Solve_h(5,28,20);       // Values from example 2
    h=Solve_h(10,15,11);      // Values from example 3
    L=Solve_L(6,4,30);        // Values from example 4
    return 0;
}
```

You can download the sources of the example here