Correction to Theorem 2 in [1]

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In the proof of Theorem 2 in [1] there is a mistake in the expression for the partial derivative of the function $J(S_o, S_c)$ with respect to $s^o_k$. As a result, the values for $\hat{s}^o_k$ and $\hat{s}^c_l$ as computed by (9)-(10) do not always achieve the (global) optimum of $J(S_o, S_c)$. It can indeed be shown that if the values of $\hat{\mu}_{ij}$ are exact, i.e. if $\hat{\mu}_{ij} = \mu_{ij} = s^o_i s^c_j$ where $s^o_i$ and $s^c_j$ are the true values of the faults, then equations (9)-(10) establish the relations $\hat{s}^o_i = s^o_i$ and $\hat{s}^c_j = s^c_j$. However, if the estimates $\hat{\mu}_{ij}$ are imprecise, then formulas (9)-(10) result in fault estimates that are slightly misplaced from the optimum. The result below is a corrected version of Theorem 2.

**Theorem 2.** Define

$$\mu = \begin{bmatrix} \hat{\mu}_{11} & \hat{\mu}_{12} & \cdots & \hat{\mu}_{1m} \\ \hat{\mu}_{21} & \hat{\mu}_{22} & \cdots & \hat{\mu}_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ \hat{\mu}_{l1} & \hat{\mu}_{l2} & \cdots & \hat{\mu}_{lm} \end{bmatrix}, \quad V_o = \begin{bmatrix} s^o_1 \\ s^o_2 \\ \vdots \\ s^o_l \end{bmatrix}, \quad V_c = \begin{bmatrix} s^c_1 \\ s^c_2 \\ \vdots \\ s^c_m \end{bmatrix},$$

and consider the optimization problem

$$\min_{V_o, V_c} J(V_o, V_c) = \|\text{vec}(V_o V_c^T) - \mu\|_F^2.$$  \hspace{1cm} (1)

Set the singular value decomposition of $\mu$ be denoted as

$$\mu = U \Sigma V^T, \quad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_{r-1} \end{bmatrix}, \quad \Sigma_i = \text{diag}(\sigma_1, \ldots, \sigma_r) > 0,$$  \hspace{1cm} (2)

with $\sigma_i \geq \sigma_{i+1}$, $i = 1, 2, \ldots, r - 1$. Then the global optimum $\{\hat{V}_c, \hat{V}_o\}$ is given by

$$\hat{V}_c = \alpha V_1, \quad \hat{V}_o = \frac{\sigma_1}{\alpha} U_1,$$  \hspace{1cm} (3)

where $\alpha \neq 0$ is a free parameter, and $U_1$ and $V_1$ are the first columns of the matrices $U$ and $V$. Moreover

$$J(\hat{V}_o, \hat{V}_c) = \sum_{i=2}^{r} \sigma_i^2.$$  \hspace{1cm} (4)

**Proof.** Note that

$$J(S_o, S_c) = \sum_{i=1}^{l} \sum_{j=1}^{m} (s^o_i s^c_j - \hat{\mu}_{ij})^2.$$
The partial derivatives of $J(V_o, V_c)$ are given by
\[
\begin{align*}
\frac{1}{2} \frac{\partial J}{\partial s_k} (V_o, V_c) &= \sum_{i=1}^l (s_i^k s_{\hat{k}}^i - \mu_{ik}) s_{\hat{i}}^o, \\
\frac{1}{2} \frac{\partial J}{\partial s_o} (V_o, V_c) &= \sum_{j=1}^r (s_j^o s_{\hat{o}}^j - \mu_{oj}) s_{\hat{c}}^c.
\end{align*}
\]

At each local optimum point $\{\hat{V}_o, \hat{V}_c\}$ it should thus hold that
\[
\begin{align*}
(\delta_k^k V_o^T - [\mu_{1k}, \mu_{2k}, \mu_{3k}])\hat{V}_o &= 0, \\
(\delta_o^k V_c^T - [\mu_{1o}, \mu_{2o}, \mu_{3o}])\hat{V}_c &= 0,
\end{align*}
\]
which written for $k = 1, 2, \ldots, m$ and $i = 1, 2, \ldots, l$ becomes
\[
\begin{align*}
\{ (\hat{V}_o \hat{V}_o^T - \mu^T)\hat{V}_o = 0, \\
(\hat{V}_o \hat{V}_c^T - \mu)\hat{V}_c = 0.
\end{align*}
\]

Using the notation in (2) we can write
\[
\begin{align*}
\{ (V_o \hat{V}_o^T - \mu^T)\hat{V}_o = 0, \\
(\hat{V}_o \hat{V}_c^T - \mu)\hat{V}_c = 0. \quad \Leftrightarrow \quad \{ V_o (V_o \hat{V}_o^T - \mu^T)U^T \hat{V}_o = 0, \\
U (V_o \hat{V}_c^T - \mu)V^T \hat{V}_c = 0,
\end{align*}
\]
so that substituting
\[
\begin{align*}
\hat{V}_c = V^T \hat{V}_c, \quad \hat{V}_o = U^T \hat{V}_o,
\end{align*}
\]
we obtain
\[
\begin{align*}
\{ (V_o \hat{V}_o^T - \Sigma^T)\hat{V}_o = 0, \\
(\hat{V}_o \hat{V}_c^T - \Sigma)\hat{V}_c = 0. \quad \Leftrightarrow \quad \{ \hat{V}_o \|\hat{V}_o\|^2_2 - \Sigma^T \hat{V}_o = 0, \\
\hat{V}_c \|\hat{V}_c\|^2_2 - \Sigma \hat{V}_c = 0, \quad \Leftrightarrow \quad \{ \Sigma \Sigma^T \hat{V}_o = \|\hat{V}_o\|^2_2 \Sigma \hat{V}_o, \\
(\Sigma^T \Sigma) \hat{V}_c = \|\hat{V}_c\|^2_2 \Sigma \hat{V}_c,
\end{align*}
\]
Therefore
\[
\begin{align*}
\Sigma \Sigma^T \hat{V}_o &= \|\hat{V}_o\|^2_2 \Sigma \hat{V}_o, \\
(\Sigma^T \Sigma) \hat{V}_c &= \|\hat{V}_c\|^2_2 \Sigma \hat{V}_c,
\end{align*}
\]
Now, let $v_{ri}$ denote a vector of length $r$, the $i$-th entry of which is equal to $1$ and all other entries are zero. Then for any scalar $\alpha_i \neq 0$ it can be written
\[
\Sigma_i^2 (\alpha_i v_{ri}) = \sigma_i^2 (\alpha_i v_{ri}), \ i = 1, 2, \ldots, r.
\]
Then all pairs $\{\hat{V}_c(i), \hat{V}_o(i)\}$ for which (6) holds are such that for $i = 1, 2, \ldots, r$
\[
\begin{align*}
\hat{V}_c(i) &= \alpha_i v_{ri}, \\
\hat{V}_o(i) &= \beta_i v_{ri}, \\
\|V_{c(i)}\|^2_2 \|V_{o(i)}\|^2_2 &= \sigma_i^2,
\end{align*}
\]
It thus follows that $\beta_i = \sigma_i / \alpha_i$. Application of the transformation (5) then gives the following expressions for all local optima
\[
\begin{align*}
\hat{V}_c(i) &= \alpha_i V_i, \\
\hat{V}_o(i) &= \frac{\sigma_i}{\alpha_i} U_i,
\end{align*}
\]
where $U_i$ and $V_i$ are the $i$-th columns of the matrices $U$ and $V$.

Now, at the $i$-th local optimum the value of the cost function can be computed as follows
\[
\begin{align*}
J(\hat{V}_o(i), \hat{V}_c(i)) &= \|\hat{V}_o(i) (\hat{V}_c(i))^T - \mu\|^2_2 = \|U_i \sigma_i V_i^T - \mu\|^2_2, \\
&= \|U_i (U^T U_i \sigma_i V_i^T V - \Sigma)V^T\|^2_2 = \|v_{ri} \sigma_i v_{ri}^T - \Sigma\|^2_2 \\
&= \sum_{j \neq i} \sigma_j^2.
\end{align*}
\]
Since $\sigma_i \geq \sigma_{i+1}$, the global optimum is achieved for $i = 1$ which completes the proof. □
Remark 1. The free scaling $\alpha$ can be found on the basis of the knowledge of at least one element of $V_o$ or one element of $V_c$.

Note also, that if the values of $\hat{\mu}_{ij}$ are exact, i.e. if $\hat{\mu}_{ij} = \mu_{ij} = s_i^o s_j^c$ where $s_i^o$ and $s_j^c$ are the true values of the faults, then the rank of $\mu$ is one, implying that the cost function becomes equal to zero at the optimum.

References