A SIMPLE PROOF OF THE MODULAR IDENTITY FOR THETA FUNCTIONS

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To A.C.M. van Rooij on occasion of his 65th birthday

Abstract. The modular identity arises in the theory of theta functions in one complex variable. It states a relation between theta functions for parameters τ and $-1/\tau$ situated in the complex upper half plane. A standard proof uses Poisson summation and hence builds on results from Fourier theory. This paper presents a simple proof using only a uniqueness property and the heat equation.

1. The θ function

Let $\mathbb{H} \subset \mathbb{C}$ denote the upper half plane of all complex numbers with a positive imaginary part. The following series converges locally uniformly in $z \in \mathbb{C}$ and $\tau \in \mathbb{H}$ and hence defines a holomorphic function on $\mathbb{C} \times \mathbb{H}$:

$$\theta(z,\tau) = \sum_{k\in\mathbb{Z}} e^{2\pi \mathrm{i} kz + \pi \mathrm{i} k^2 \tau}$$

This function is often called the θ_3 function of Jacobi (some texts use $q = e^{\pi \mathrm{i}\tau}$ or replace πz by z). It satisfies the shift relations in z

$$\theta(z+1,\tau) = \theta(z,\tau) \tag{1.1}$$

and

$$\theta(z+\tau,\tau) = e^{-2\pi \mathrm{i} z - \pi \mathrm{i}\tau}\theta(z,\tau) \tag{1.2}$$

that can easily be verified from its definition. The following *heat equation* is also apparent from the definition of θ:

$$\frac{d^2\theta}{dz^2} = 4\pi \mathrm{i}\frac{d\theta}{d\tau}. \tag{1.3}$$

Let $\Lambda(\tau) = \mathbb{Z} + \mathbb{Z}\tau$ be the lattice spanned by 1 and τ. For fixed parameter τ the function θ in z is the only entire function satisfying (1.1) and (1.2) up to complex multiples. This follows from the following theorem:

Theorem 1.1. *If $f(z)$ is an entire function on $\mathbb{C}$ satisfying the shift relations (1.1) and (1.2) then either f vanishes identically or all its roots equal $(\tau+1)/2$ modulo the lattice $\Lambda(\tau)$.*

Suppose f does not vanish identically. The shift relations for f imply

$$\frac{f'(z+1)}{f(z+1)} = \frac{f'(z)}{f(z)} \quad\text{and}\quad \frac{f'(z+\tau)}{f(z+\tau)} = \frac{f'(z)}{f(z)} - 2\pi \mathrm{i}.$$

For $b \in \mathbb{C}$ define a closed fundamental domain $P \subset \mathbb{C}$ by

$$P = \{b + x + y\tau \mid x, y \in [0,1]\}.$$

The number of roots of f on P and the sum of its roots on P can be computed by the integrals

$$\frac{1}{2\pi \mathrm{i}} \oint_{\partial P} \frac{f'(z)}{f(z)} dz$$

and

$$\frac{1}{2\pi \mathrm{i}} \oint_{\partial P} \frac{z f'(z)}{f(z)} dz$$

respectively. By varying the number b we may assume that f has no roots on ∂P so both integrals are well defined. Using the shift relations for f shows that the first integral evaluates to 1, so f has only one root on P. The second integral evaluates to a value equal to $(\tau+1)/2$ modulo the lattice $\Lambda(\tau)$. This proves the theorem. □

Corollary 1.2. *If f is as theorem 1.1, then $f(z) = c \cdot \theta(z, \tau)$ for some constant $c \in \mathbb{C}$.*

By theorem 1.1 we find that $\theta(0,\tau) f(z) - f(0)\theta(z,\tau)$ must vanish identically as it vanishes at $z = 0$ as well as at $(\tau+1)/2$. □

2. THE MODULAR IDENTITY

We are already in a position to prove the modular identity (2.3) for θ. A very accessible treatment of this identity using Poisson summation can be found in [1]. For the proof given below, the heat equation suffices. In [2] (in particular pages 69–70) a similar approach is taken, based on Fourier expansion in the parameter z rather than contour integration as in the previous section. Define an entire function ϑ by

$$\vartheta(z) = e^{\pi \mathrm{i} \tau z^2} \theta(\tau z, \tau).$$

Then $\vartheta(z+1) = \vartheta(z)$ and

$$\vartheta(z - 1/\tau) = e^{-2\pi \mathrm{i} z + \pi \mathrm{i}/\tau} \vartheta(z).$$

Hence $\vartheta(z) = c(\tau) \cdot \theta(z, -1/\tau)$ for some function c on the upper half plane by corollary 1.2. Substituting $\tau = \mathrm{i}$ and $z = 0$ shows that $c(\mathrm{i}) = 1$. The heat equation for θ will produce a simple differential equation for c. Elementary computations show:

$$\frac{d^2\vartheta}{dz^2}(0) = 2\pi \mathrm{i} \tau \theta(0,\tau) + \tau^2 \frac{d^2\theta}{dz^2}(0,\tau) = c(\tau) \frac{d^2\theta}{dz^2}(0, -1/\tau) \tag{2.1}$$

$$\frac{d\vartheta}{d\tau}(0) = \frac{d\theta}{d\tau}(0,\tau) = c'(\tau)\theta(0,-1/\tau) + c(\tau)\tau^{-2} \frac{d\theta}{d\tau}(0,-1/\tau). \tag{2.2}$$

Using the heat equation (1.3) on (2.1) yields

$$\tfrac{1}{2}\tau^{-1}\theta(0,\tau) + \frac{d\theta}{d\tau}(0,\tau) = c(\tau)\tau^{-2}\frac{d\theta}{d\tau}(0,-1/\tau)$$

and combining this with (2.2) leads to

$$\theta(0,\tau) = -2\tau c'(\tau)\theta(0,-1/\tau).$$

However, substituting $z = 0$ in $\vartheta(z)$ gives

$$\theta(0,\tau) = \vartheta(0) = c(\tau)\theta(0,-1/\tau)$$

and as θ does not vanish at $z = 0$ we find

$$-2\tau c'(\tau) = c(\tau).$$

Together with $c(\mathrm{i}) = 1$ we finally find

$$c(\tau) = \frac{1}{\sqrt{-\mathrm{i}\tau}}$$

and thus the modular identity for the θ function:

$$\theta(z, -1/\tau) = \sqrt{-\mathrm{i}\tau} e^{\pi \mathrm{i} \tau z^2} \theta(\tau z, \tau). \tag{2.3}$$

References

[1] R. Bellman. *A brief introduction to theta functions*. Holt, Rinehart and Winston, New York, 1961.

[2] L. Ehrenpreis. Fourier analysis, partial differential equations and automorphic functions. In L. Ehrenpreis and R. C. Gunning, editors, *Theta Functions Bowdoin 1987*, volume 49–Part 2 of *Proceedings of Symposia in Pure Mathematics*, pages 45–100. AMS, 1989.